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Homology of Fortuin–Kasteleyn Clusters of Potts Models on the Torus

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Topological properties of Fortuin–Kasteleyn clusters are studied on the torus. Namely, the probability that their topology yields a given subgroup of the first homology group of the torus is computed for $Q=1$, 2, 3 and 4. The expressions generalize those obtained by Pinson for percolation ($Q=1$). Numerical results are also presented for three tori of different moduli. They agree with the theoretical predictions for $Q=1$, 2 and 3. For $Q=4$ agreement is not ruled out but logarithmic corrections are probably present and they make it harder to decide.

KEY WORDS: Potts models; bond percolation; torus.

1. INTRODUCTION

The Fortuin–Kasteleyn (FK) formulation of Potts models is often referred to as the Q-state bond-correlated percolation. The clusters formed by sites and bonds, called FK clusters, may wrap the torus in a non-trivial way. One can study the particular topology of a cluster by looking at its first homology group. This group will be isomorphic to a subgroup of the homology group of the surface on which the model is defined. In general, consider $\mathcal{G}$, the topological space given by all FK clusters of a configuration imbedded in a Riemann surface S. There is a natural homomorphism $\phi: H_1(\mathcal{G}) \to H_1(S)$ between first homology group of each space. Therefore one may look for the probability that $\phi(H_1(\mathcal{G}))$ be a given subgroup G of $H_1(S)$. A definition of this observable was first given by Langlands *et al.* in ref. 1. In a preceding work, Pinson computed these probabilities in the percolation case on the torus using a method introduced by di Francesco *et al.*[2,3]

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This paper generalizes Pinson's work on percolation to Potts models on the torus for $Q=1$, 2, 3 and 4 to compute the probability that a FK configuration of a Q-state Potts model yields a particular subgroup of the first homology group of the torus. This observable is also studied numerically using Monte-Carlo simulations.

The general concepts required for our calculation are defined in Section 2. Then essential points on the transformation from the Potts model to a gaussian free field are discussed before we deduce the explicit expressions for probabilities in Section 4. As expected, these are found to be modular invariant. Numerical results presented for three tori with different ratios are in good agreement with values computed except for $Q=4$. For the latter case, it is difficult to estimate the continuum limit numerically because what looks like logarithmic finite-size corrections seem to appear.

2. DEFINITIONS

A FK configuration (or FK subgraph) $\mathscr{G}$ is built on a Potts configuration σ by putting bonds between all same-state neighbors and then removing them with probability $1-p \equiv 1-\exp(-K)$ where K is the critical ratio $J/k_B T_c$. The resulting graph $\mathscr{G}$ is said a FK subgraph associated to σ. Neither correspondences $\sigma \to \mathscr{G}$ nor $\mathscr{G} \to \sigma$ are unique. A maximally connected component in $\mathscr{G}$ is called a FK cluster.

In this paper, we define the model on the torus, i.e., $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, $\tau \in \mathbb{Z}$. The first homology group of the torus T is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Therefore its subgroups are isomorphic to $\{0\}$, $\mathbb{Z} \times \mathbb{Z}$ and $\{m, n\} \equiv \langle (m, n) \rangle$ where m, $n \in \mathbb{Z}$.

Let α and β be two independent cycles on the torus. On the parallelogram 0, τ, 1, $1+\tau$, we choose α to be horizontal and β to be parallel to the edge $(0, \tau)$. We distinguish different classes of FK clusters. They can be homotopic to a point. They may wind the torus a times along α and b times along β, where $a, b \in \mathbb{Z}$ and $a \wedge b \equiv \gcd(a, b)$ must be one. A cluster of this type will be called a cluster $\{a, b\}$. We define the signs of a and b as follows. Let us say we draw a boundary curve of such a cluster. Then $a>0$ if we go from left to right and $b>0$ if we go from top to bottom. Hence a cluster $\{a, b\}$ is equivalent to a cluster $\{-a, -b\}$. The last class is formed by cluster with a cross topology, precisely clusters containing at least two connected subcomponents with different non-trivial windings $\{a, b\}$.

We consider the natural homomorphism ϕ: $H_1(\mathscr{G}) \to H_1(T)$ for a FK subgraph $\mathscr{G}$ and the torus T. We are interested in the subgroup G of $\mathbb{Z} \times \mathbb{Z}$ which lies in $\phi(H_1(\mathscr{G}))$. Then, we associate to $\mathscr{G}$ this particular subgroup G. In fact, the correspondence is:

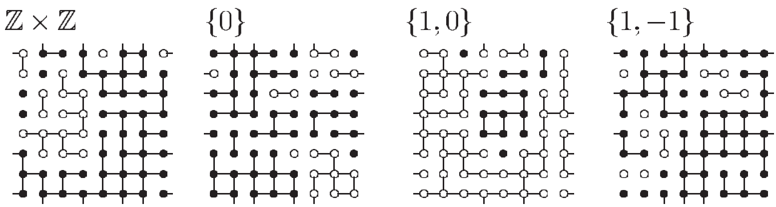

Fig. 1. Different $Q=2$ configurations with their associated subgroup.

- $G=\{0\}$ if $\mathcal{G}$ contains only clusters homotopic to a point;
- $G=\{a,b\}$ if $\mathcal{G}$ has only clusters homotopic to a point and at least one cluster of class $\{a,b\}$;
- $G=\mathbb{Z}\times\mathbb{Z}$ if $\mathcal{G}$ contains a cross.

Some examples are given in Fig. 1 for $Q=2$. The two states are depicted respectively by black dots and white dots. Bonds were added between same-state neighbors with probability p. The subgraphs associated with each subgroup will be respectively called: trivial subgraphs, subgraphs $\{a,b\}$ and subgraphs with a cross topology. Because of the nature of FK clusters the probability of $G=\{a,b\}$, $a\wedge b\neq 1$ is zero. Hence we will assume in the rest of the paper that $a\wedge b=1$.

Let $Z_Q(G)$ be the partition function restricted to FK configurations associated to the subgroup G in the continuum limit. The probability that a subgraph yields G is then

$$\pi_Q(G)=\frac{Z_Q(G)}{Z_Q} \tag{1}$$

where Z_Q is $Z_Q(\{0\})+Z_Q(\mathbb{Z}\times\mathbb{Z})+\sum_{a\wedge b=1} Z_Q(\{a,b\})$.

3. FORTUIN–KASTELEYN CLUSTERS AND GAUSSIAN FREE FIELDS

To compute each weight, a tranformation of the Potts model in the FK formulation to the F model (or six-vertex model) and finally to the SOS model will be performed. The renormalization of the SOS model into a bosonic free field is then used to get the probabilities in the continuum limit.

The first step of the transformation involves a decomposition of subgraphs into oriented-polygon decompositions. There is a bijection between subgraphs and possible non-oriented polygon decompositions on a lattice of $M \times N$ sites.[4] The partition function of the Potts model in the FK formulation can then be rewritten as a sum of separated weights at the critical temperature:[2, 3]

$$\mathscr{Z}_Q^{M\times N} = \mathscr{Z}_Q^{M\times N}(\{0\}) + \mathscr{Z}_Q^{M\times N}(\mathbb{Z}\times\mathbb{Z}) + \sum_{a\wedge b=1} \mathscr{Z}_Q^{M\times N}(\{a,b\})$$

$$= \sum_{\mathscr{P}\{0\}} Q^{N_p/2} + Q \sum_{\mathscr{P}\{\mathbb{Z}\times\mathbb{Z}\}} Q^{N_p/2} + \sum_{a\wedge b=1} \sum_{\mathscr{P}\{a,b\}} Q^{N_p/2}. \tag{2}$$

where $\mathscr{P}\{G\}$ stands for all decompositions whose subgraph generates G. Here, N_p stands for the number of polygons in a specific decomposition. The Q factor in the restricted weight of $\mathbb{Z}\times\mathbb{Z}$ comes from a different Euler relation between the number of sites, polygons, connected components and faces for connected components with a cross topology.

One can build a bijective correspondence between trivial subgraphs and subgraphs with a cross topology which preserves the number of polygons in the polygon decomposition. Let $\mathscr{G}_+$ be a subgraph of the cross topology type with decomposition $\mathscr{P}$. A trivial subgraph $\mathscr{G}_0$ is built on the dual lattice of $\mathscr{G}_+$ keeping the same $\mathscr{P}$ by putting its sites in the middle of each smallest square formed by four sites of $\mathscr{G}_+$. Then, on the dual lattice (sites in the middle of plaquettes), a subgraph $\mathscr{G}_0$ is constructed by putting all bonds that do not intersect with bonds of $\mathscr{G}_+$ on the original lattice. The correspondence is bijective by construction and $\mathscr{G}_0$ is necessarily trivial. One can also see that $\mathscr{G}_+$ and $\mathscr{G}_0$ have the same number of polygons. Consequently, we obtain an important duality relation for weights of $\{0\}$ and $\mathbb{Z}\times\mathbb{Z}$ in (2):

$$\mathscr{Z}_Q^{M\times N}(\mathbb{Z}\times\mathbb{Z}) = Q\mathscr{Z}_Q^{M\times N}(\{0\}). \tag{3}$$

In the F model, we express the factor $Q^{N_p/2}$ in terms of a random orientation on each polygon. Let us write $\epsilon_i = \pm 1$ for non-trivial polygons (type n-h) and $\epsilon_j = \pm 1$ for polygons homotopic to a point (type h). We impose:

$$Q^{N_p/2} = Q^{N_{n-h}/2} Q^{N_h/2} = \sum_{\{\epsilon_i=\pm 1\}} \cos\left[\frac{\pi}{2} e_0 \sum_i \epsilon_i\right] \sum_{\{\epsilon_j=\pm 1\}} \exp\left[4i\lambda \sum_j \epsilon_j\right] \tag{4}$$

where e_0 and λ are chosen such as $Q^{1/2} = 2\cos(\pi e_0/2)$ and $Q = 2\cos(4\lambda)$. We do not need the last details of the transformation to the F model for our purpose, but the reader could find them in ref. 4.

To construct the *SOS* model from the F one, a field must be defined on the faces of the lattice.[2, 3] These faces are formed by polygonal curves of type n-h or h. A polygonal line means an increase or a decrease by a factor $\pi/2$ of the field depending on the polygon orientation. Discontinuities or frustrations are therefore induced in the field by polygons *n-h* of the decomposition. For subgraphs with a cross topology and trivial one, only trivial polygons are present; thus the field is always periodic. For subgraphs $\{a, b\}$, if we write respectively $\delta\phi_1$ and $\delta\phi_\tau$ for discontinuities observed along the cycles α and β, we have :

$$\delta\phi_1 = \frac{\pi}{2} b \sum_i \epsilon_i = \pi m \qquad \delta\phi_\tau = \frac{\pi}{2} a \sum_i \epsilon_i = \pi m' \tag{5}$$

where m, $m' \in \mathbb{Z}$ and the sum is only over *n-h* polygons. For subgraphs with a cross topology and trivial one, m and m' are zero. Subgraphs $\{a, b\}$ have possibly polygons h and necessarily two polygons *n-h* of class $\pm[a, b]$ for each non-trivial connected component. Therefore, their boundary conditions will be: $m = bk$, $m' = ak$ for a certain $k \in \mathbb{Z}$. Moreover

Remark 1. because $a \wedge b = 1$, only a unique subgroup $\{a, b\}$ can produce non-periodic field with given discontinuities m, m', $(m, m') \neq 0$.

We can express the correction term due to non-trivial polygons in (4) as $\cos\left[\frac{\pi}{2} e_0 \sum_i \epsilon_i\right] = \cos[\pi e_0 (m \wedge m')]$ with the help of (5) and the fact that $a \wedge b = 1$.

The extra factor Q in front of the second term in (2) does not appear in the usual partition function of the F model. We have not taken it into account yet in the SOS model partition function. Consequently, let us first consider $\hat{\mathscr{Z}}_Q^{M \times N}$, the partition function of the Q-state Potts model with a weight Q times lower than the correct one for subgraphs with a cross topology. In regard to the transformation performed, it corresponds to the SOS model partition function with the non-trivial correction. Considering that the SOS model is assumed to renormalize onto a gaussian free field, $\hat{\mathscr{Z}}_Q^{M \times N}$ becomes in the continuum limit:[2]

$$\hat{\mathscr{Z}}_Q^{M \times N}(e_0) \to \hat{Z}(g, e_0) \equiv \sum_{m, m' \in \mathbb{Z}} Z_{m, m'}(g/4) \cos\left[\pi e_0 (m \wedge m')\right]. \tag{6}$$

The hat is the reminder of the missing factor Q. The value of g is given by $Q = 2 + 2\cos(\pi g/2)$, $g \in [2, 4]$ (see ref. 5). The term $Z_{m, m'}(g)$ is the bosonic partition function on a torus $\tau = \tau_R + i\tau_I$ for a field with discontinuity $\delta_1 \phi = 2\pi m$, $\delta_\tau \phi = 2\pi m'$ and coupling constant g:

$$Z_{m, m'}(g) = \frac{\sqrt{g}}{\tau_I{}^{1/2}\, |\eta(q)|^2} \exp\left[-\pi g \frac{m^2 \tau_I^2 + (m' - m\tau_R)^2}{\tau_I}\right] \tag{7}$$

where $q = \exp(2i\pi\tau)$ and $\eta(q)$ is the Dedekind function.

4. PROBABILITIES OF HOMOLOGY SUBGROUPS

Each specific weight in (2) can be derived using expressions of the last section. The partition function is then easily computed to get the probability (1) for a particular subgroup G. Every expression found must be modular invariant.

We first find the weights of $\{0\}$ and $\mathbb{Z}\times\mathbb{Z}$. We notice that for $e_0=1$ in (4), the quantity $Q^{N_{n-h}/2}$ is zero , therefore $Z_Q(\{a,b\})|_{e_0=1}=0$. Hence, if we use this in (6), we get:

$$\hat{Z}(g,1)=Z_Q(\{0\})+\frac{1}{Q}Z_Q(\mathbb{Z}\times\mathbb{Z}). \tag{8}$$

The desired results then follow from (3)

$$Z_Q(\{0\})=\frac{1}{2}\sum_{m,m'\in\mathbb{Z}} Z_{m,m'}(g/4)\cos\left[\pi(m\wedge m')\right] \tag{9}$$

$$Z_Q(\mathbb{Z}\times\mathbb{Z})=\frac{Q}{2}\sum_{m,m'\in\mathbb{Z}} Z_{m,m'}(g/4)\cos\left[\pi(m\wedge m')\right]. \tag{10}$$

We introduce the notation $Z^{a,b}_{0,0}(g/4)$ for the weight of all subgraphs $\{a,b\}$ which induce periodic field and write $\tilde{Z}_{m,m'}(g/4,e_0)\equiv Z_{m,m'}(g/4)\cos\left[\pi e_0(m\wedge m')\right]$ for convenience. The weight of the subgroup $\{a,b\}$ is then given by the sum over all non-zero boundary conditions particular to this subgroup by remark 1 plus the contribution of subgraphs $\{a,b\}$ with periodic field:

$$Z_Q(\{a,b\})=Z^{a,b}_{0,0}(g/4)+\sum_{\substack{m=bk\\ m'=ak\\ k\in\mathbb{Z}\setminus\{0\}}}\tilde{Z}_{m,m'}(g/4,e_0). \tag{11}$$

For $e_0=1$, it follows directly from (11) that:

$$Z^{a,b}_{0,0}(g/4)=-\sum_{\substack{m=bk\\ m'=ak\\ k\in\mathbb{Z}\setminus\{0\}}}\tilde{Z}_{m,m'}(g/4,1). \tag{12}$$

We insert this new expression for $Z^{a,b}_{0,0}(g/4)$ in (11). The correct weight is then given by:

$$Z_Q(\{a,b\})=\sum_{\substack{m=bk\\ m'=ak\\ k\in\mathbb{Z}\setminus\{0\}}}\left[\tilde{Z}_{m,m'}(g/4,e_0)-\tilde{Z}_{m,m'}(g/4,1)\right]. \tag{13}$$

The partition function is computed by summing over all subgroups:

$$Z_Q = \frac{(Q+1)}{2}\hat{Z}(g,1) + \sum_{a\wedge b=1} \sum_{\substack{m=bk\\ m'=ak\\ k\in\mathbb{Z}\setminus\{0\}}} [\tilde{Z}_{m,m'}(g/4,e_0) - \tilde{Z}_{m,m'}(g/4,1)]$$

$$= \hat{Z}(g,e_0) + \frac{(Q-1)}{2}\hat{Z}(g,1) \tag{14}$$

in agreement with ref. 2. We added zero to the r.h.s of the first equality in (14) by considering the term $m=m'=0$ in the sum.

Each probability can then be calculated with (1). For a specific Q, sums over $\tilde{Z}(g/4,e_0)$ involved in the expressions found are computed using restricted sums over equivalence class of $m\wedge m'$ so the cosinus is a constant in each of them.

The subgroups $\{0\}$ and $\mathbb{Z}\times\mathbb{Z}$ are both stable under the action of any modular transformation of the torus whose generators are $T: \tau\to\tau+1$ and $S: \tau\to -1/\tau$. Hence, their probability must be invariant. For subgraphs $\{a,b\}$, modular transformations of non-trivial curves impose:

$$\pi_Q(\{a,b\})|_\tau = \pi_Q(\{a+b,b\})|_{\tau+1} \tag{15}$$

$$\pi_Q(\{a,b\})|_\tau = \pi_Q(\{-b,a\})|_{-1/\tau}. \tag{16}$$

These restrictions are easily verified directly on (13) with the help of the modular transformation of the bosonic partition function:[6]

$$Z_{m,m'}|_\tau = Z_{m,m+m'}|_{\tau+1} \tag{17}$$

$$Z_{m,m'}|_\tau = Z_{m',-m}|_{-1/\tau}. \tag{18}$$

In the particular cases $Q=1$ and $Q=2$, the expressions for $\pi_Q(\{a,b\})$ are written in an elegant way using theta functions.[6] The calculations yield for $\tau_{a,b}\equiv a-b\tau$

$$\pi_1(\{a,b\}) = \frac{1}{2\,|\tau_{a,b}|\,|\eta(q)|^2}\left[\theta_3\left(\frac{i\tau_I}{6\,|\tau_{a,b}|^2}\right) - \theta_3\left(\frac{3i\tau_I}{2\,|\tau_{a,b}|^2}\right) - 2\theta_2\left(\frac{3i\tau_I}{2\,|\tau_{a,b}|^2}\right)\right] \tag{19}$$

$$\pi_2(\{a,b\}) = \frac{1}{|\tau_{a,b}|\,|\eta(q)|}\left[\frac{\theta_2\left(\frac{i\tau_I}{3\,|\tau_{a,b}|^2}\right) - \theta_3\left(\frac{i\tau_I}{3\,|\tau_{a,b}|^2}\right) + \theta_4\left(\frac{i\tau_I}{3\,|\tau_{a,b}|^2}\right)}{|\theta_2(\tau)| + |\theta_3(\tau)| + |\theta_4(\tau)|}\right]. \tag{20}$$

5. NUMERICAL SIMULATIONS

Numerical simulations were performed on square lattices with $V \times H$ sites on three different tori, $\tau = i$, $\tau = 2i$ and $\tau = i + 1/2$. We chose to consider the quantity $\pi_Q(\{0\}) + \pi_Q(\mathbb{Z} \times \mathbb{Z})$ instead of two different values because they are simply related by (3) for every lattice. This allowed us technically to experiment on larger samples. Results are shown in Table 1.

On a $V \times H$ lattice, the identification rules on the boundaries for a site (i, j) were $(i, H) \equiv (i, 0)$, $(V, j) \equiv (0, j)$ for $\tau = i$ and $\tau = 2i$ and $(i, H) \equiv (i, 0)$, $(V, j) = (0, j + V/2 \bmod V)$ for $\tau = i + 1/2$. We used Swendsen–Wang algorithm for $Q = 2$, 3 and 4 with at least 8000 initial thermalisations and five upgrades between measurements.

We proceed differently to approximate the continuum limit for $Q = 1$, 2 than for $Q = 3$, 4. In the first case, we took lattices of dimensions 256×256 sites for $\tau = i$ and $\tau = i + 1/2$, 256×128 for $\tau = 2i$. These are indeed good approximations of the continuum limit as can be seen by comparing

Table I. Probabilities Calculated with Both Analytical and Numerical Method

Q	τ	$\pi_Q(\{0\}) + \pi_Q(\mathbb{Z} \times \mathbb{Z})$	$\pi_Q(\{1, 0\})$	$\pi_Q(\{0, 1\})$	$\pi_Q(\{1, 1\})$	$\pi_Q(\{1, -1\})$
1	i	**0.61908**	**0.16942**	**0.16942**	**0.020980**	**0.020980**
		0.61903|22	0.16948|17	0.16939|17	0.020995|64	0.020957|64
	$2i$	**0.45376**	**0.50304**	**0.024950**	**0.008755**	**0.008755**
		0.45380|22	0.50293|22	0.024983|70	0.008749|42	0.008779|42
	$i+1/2$	**0.62840**	**0.16816**	**0.10006**	**0.10006**	**0.001519**
		0.62849|22	0.16812|17	0.10008|13	0.09998|13	0.001504|17
2	i	**0.67927**	**0.14644**	**0.14644**	**0.013903**	**0.013903**
		0.67907|21	0.14654|16	0.14645|16	0.013949|53	0.013940|53
	$2i$	**0.47863**	**0.49625**	**0.015376**	**0.004734**	**0.004734**
		0.47888|32	0.49595|32	0.015384|78	0.004751|43	0.004760|43
	$i+1/2$	**0.69083**	**0.14569**	**0.08095**	**0.08095**	**0.000727**
		0.69091|22	0.14569|17	0.08100|13	0.08081|13	0.000724|13
3	i	**0.74533**	**0.11867**	**0.11867**	**0.008660**	**0.008660**
		0.7457|21	0.1173|19	0.1187|14	0.00863|42	0.00871|49
	$2i$	**0.52571**	**0.45995**	**0.0092445**	**0.0024968**	**0.0024968**
		0.5250|21	0.4605|21	0.00921|45	0.00240|29	0.00247|23
	$i+1/2$	**0.75821**	**0.11820**	**0.061448**	**0.061448**	**0.00033**
		0.7584|19	0.1180|15	0.0606|13	0.0608|13	0.00000|81
4	i	**0.85034**	**0.071728**	**0.071728**	**0.0030997**	**0.0030997**
		0.8265|23	0.08343|16	0.08151|16	0.00380|43	0.00440|32
	$2i$	**0.63368**	**0.36172**	**0.0032494**	**0.00067549**	**0.00067549**
		0.6013|25	0.3929|25	0.00369|28	0.00087|16	0.00085|14
	$i+1/2$	**0.86316**	**0.07151**	**0.03260**	**0.03260**	**0.000061**
		0.8426|23	0.0794|19	0.0389|11	0.0383|13	0.00011|4

measurements on lattices with half this linear size. In Table 1, the numbers after the vertical bar represent the statistical error on the last two digits for a 95%-confidence interval. For example, 0.0210|10 means the interval [0.0200, 0.220]. Each probability was calculated on a sample larger than 2×10^7 configurations. The figures in bold faces are the predictions obtained from (9), (10) and (13). The agreement between numerical data and analytic calculations are convincing.

For $Q = 3$ and $Q = 4$, a more complex method is necessary because the scale effect is still visible in large lattices. Consequently, we used a power law hypothesis:(7)

$$|\pi_Q(G, H, V) - \pi_Q(G)| \sim \alpha V^{\beta}. \tag{21}$$

where $\alpha > 0$ and $\beta < 0$ are two parameters to be determined. We obtained α and β by a linear regression of the function $f[i] \equiv \log |\pi_Q(\{a, b\}, 2^i H_0, 2^i V_0) - \pi_Q(\{a, b\}, 2^{i+1} H_0, 2^{i+1} V_0)|$. We took $V_0 = 16$ for $Q = 3$ and $V_0 = 32$

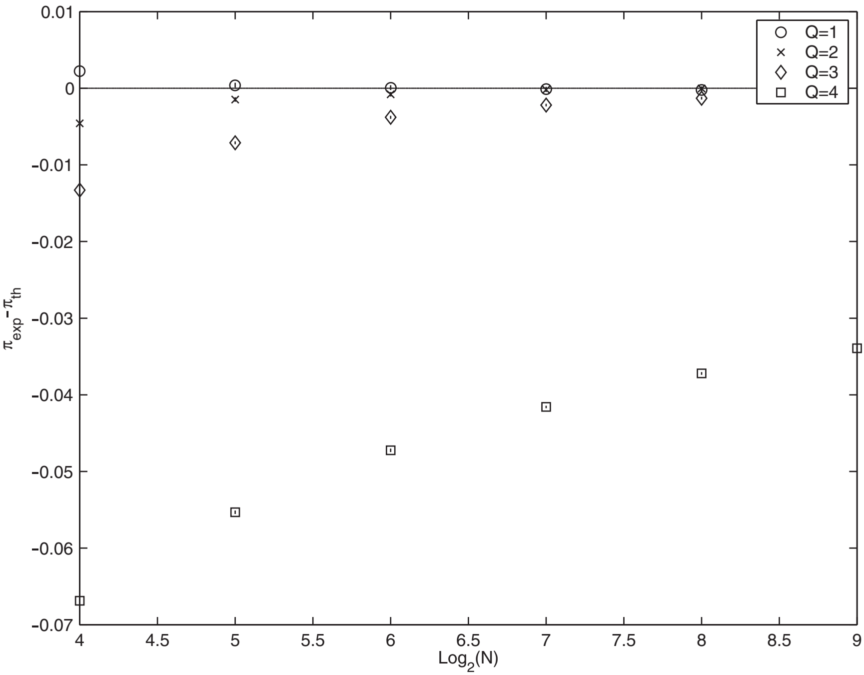

Fig. 2. Deviation of numerical results from theoretical prediction for $\pi(\mathbb{Z} \times \mathbb{Z}) + \pi(\{0\})$ on a $N \times N$ lattice ($\tau = i$) as a function of $\log_2(N)$

for $Q = 4$ with $i = 0, 1, 2, 3, 4$. The errors were estimated using a numerical method. For each i, we had a sample of at least 2×10^7 configurations. The results are very good for $Q = 3$ although errors are larger using this method. One might ask whether the exponent β of (21) is universal. However, the values of the exponent β, say in the case $Q = 3$, differed for the different subgroups and the different ratios studied though they remained of order 1. For $Q = 4$, we naively tried the same approximation. These results in Table 1 and the behaviour plotted in Fig. 2 suggest that the finite-size correction term is not of the form (21). Several authors have argued convincingly that finite-size effects for the $Q = 4$ Potts model show logarithmic corrections.[8] It is hard to see them in our data since the fit would involve more parameters and thus would need more points to be accurate. Thus one would have to experiment on lattices larger than 512×512 and to deal with the problem of getting samples with a relatively small statistical error.

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